

All investors are risk averse expected utility maximizers

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Abstract

Assuming that agents' preferences satisfy first-order stochastic dominance, we show that the Expected Utility Paradigm can explain all rational investment choices. In particular, the optimal investment strategy in any behavioral law-invariant setting corresponds to the optimum for some risk averse expected utility maximizer whose concave utility function we derive explicitly. This result enables us to infer agents' utility and risk aversion from their investment choice in a non-parametric way. We also show that decreasing absolute risk aversion (DARA) is equivalent to a demand for terminal wealth that has more spread than the opposite of the log pricing kernel at the investment horizon.

Key-words: First-order stochastic dominance, Expected Utility, Utility Estimation, Risk Aversion, Law-invariant Preferences, Decreasing Absolute Risk Aversion, Arrow-Pratt risk aversion measure.

JEL codes: G11, D03, D11, G02.

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1 Introduction

First suggested by Cramer and Bernoulli (1738) and rigorously introduced in the economic literature by von Neumann and Morgenstern (1947), Expected Utility Theory (EUT) has for decades been the dominant theory for making decisions under risk. Yet, it has been criticized for not always being consistent with agents' observed behavior (e.g. paradox of Allais (1953), Starmer (2000)). In response to this, numerous alternatives have been proposed, most notably the dual theory (Yaari (1987)), the rank-dependent utility theory (Quiggin (1993)) and the cumulative prospect theory (Tversky and Kahneman (1992)). These competing theories differ significantly but usually satisfy first-order stochastic dominance (FSD). Many economists consider a violation of this property as grounds for refuting a particular model; see for example Birnbaum (1997), Birnbaum and Navarrette (1998), and also Levy (2008) for more discussions and empirical evidence of FSD violations. Recall also that although prospect theory (Kahneman and Tversky (1979)) provides explanations for phenomena that were unexplained before, it violates first-order stochastic dominance. To overcome this potential issue, Tversky and Kahneman (1992) have developed Cumulative Prospect Theory¹.

In this paper, we show that the optimal portfolio in any behavioral theory which respects first-order stochastic dominance (FSD) can be obtained as an optimal portfolio for a risk averse expected utility maximizer. In particular, for any agent behaving according to the cumulative prospect theory (CPT investor), there is a corresponding risk averse expected utility maximizer purchasing the same optimal portfolio, even if the CPT investor can exhibit risk seeking behavior for losses. This is not meant to be a criticism of standard theories as they have been developed mainly in the context of “a comparison between gambles” and are not rooted in portfolio selection.

The proof of our main results uses Dybvig's (1988a), (1988b) alternative setting for portfolio selection. Instead of optimizing a value function, Dybvig (1988a) specifies a target distribution and solves for the strategy that generates the distribution at the lowest possible cost. It may indeed be more natural for an investor to describe her target distribution of terminal wealth² instead of her utility function. Recall also that the pioneering work

¹See also Barberis and Huang (2008), Bernard and Ghossoub (2010), Chateauneuf and Wakker (1999), De Giorgi and Hens (2006), He and Zhou (2011a), Levy and Levy (2002) for more studies and extensions of the original paper of Kahneman and Tversky (1979) and recent developments in behavioral finance.

²For example, Goldstein, Johnson and Sharpe (2008) discuss how to estimate the dis-

in portfolio selection by Markowitz (1952) and Roy (1952) is solely based on the mean and variance of returns and does not invoke utility functions³.

For every rational investor in a theory which respects FSD (which is shown to be equivalent to law-invariant and non-decreasing preferences), we exhibit a risk averse EUT optimizer with the same optimal consumption. This link between EUT and a possibly non-expected utility setting can be used to estimate agents' utility functions and risk aversion coefficients. Our approach is non-parametric and is solely based on the knowledge of the distribution of optimal wealth and of the financial market. This is in contrast to traditional approaches for inferring utility and risk aversion: one typically starts by specifying an exogenous parametric utility function in isolation of the market in which the agent invests and then calibrates this utility function using questionnaires, laboratory experiments and econometric analysis of panel data. Inference of risk preferences from observed investment behavior has also been studied by Sharpe (2007) and Sharpe, Goldstein and Blythe (2000), Dybvig and Rogers (1997) and Musiela and Zariphopoulos (2010)⁴. Sharpe (2007) and Sharpe, Goldstein and Blythe (2000) assume a static setting and rely on Dybvig's (1988a), (1988b) results to estimate the coefficient of constant relative risk aversion for a CRRA utility based on target distributions of final wealth. Here, we propose a non-parametric method to construct for every prescribed final wealth distribution, an explicit utility function that generates the prescribed final wealth distribution if the EUT investor maximizes her expected utility using the constructed utility function. This then allows us to infer the investor's risk aversion coefficients.

It is widely accepted that the Arrow-Pratt measure of absolute risk aversion is decreasing (DARA) with wealth - this is often the motivation for using the CRRA utility instead of the exponential one. We show that the DARA property is characterized by a demand for final wealth W that has more spread than the opposite of the log pricing kernel, H . Precisely, let W and H be distributed with F_W and F_H respectively. Then, the Arrow-Pratt measure for absolute risk aversion is decreasing if and only if $F_W^{-1}(F_H(x))$

tribution at retirement using a questionnaire.

³Black (1988) calls a utility function "a foreign concept for most individuals" and also states that "instead of specifying his preferences among various gambles the individual can specify his consumption function".

⁴Under some conditions, Dybvig and Rogers (1997) and Musiela and Zariphopoulos (2010) infer utility from dynamic investment decisions. Our setting is static and well adapted to the investment practice where consumers purchase a financial contract and do not trade afterwards.

is strictly convex. This extends recent work by Dybvig and Wang (2012) in another direction. They show that if agent A has lower absolute risk aversion than agent B , then the terminal wealth for agent A is distributed as the other's terminal wealth plus a nonnegative random variable plus conditional-mean-zero noise; see also and Beiglböck, Muhle-Karbe and Temme (2012).

The paper is organized as follows. Section 2 describes the setting and highlights the strong connection between law-invariance and first-order stochastic dominance. Section 3 contains our main results and shows that any distribution for terminal wealth can be obtained as the optimum of an expected utility maximizer with non-decreasing and (possibly non strictly) concave utility. Section 4 gives some application of the main results derived in Section 3. In particular, we illustrate how a non-decreasing concave utility function can be constructed to explain the demand for optimal investments in Yaari's (1987) setting. In Section 5, we show how to derive the coefficients for risk aversion directly from the choice for the distribution of final wealth and the financial market. We also explore the precise connections between decreasing risk aversion and variability of terminal wealth. In Section 6, we derive new utilities corresponding to well-known distributions and discuss their properties. Finally, Section 7 concludes.

2 Setting

We assume an arbitrage-free and frictionless financial market $(\Omega, \mathcal{F}, \mathbb{P})$ with a fixed investment horizon of $T > 0$. Let ξ_T be the pricing kernel that is agreed upon by all agents. We assume it has a density on $\mathbb{R}^+ \setminus \{0\}$. The value ω_0 at time 0 of a consumption X_T at T is then computed as,

$$\omega_0 = E[\xi_T X_T].$$

We only consider terminal consumptions X_T such that ω_0 is finite. In the sequel, we always consider agents with law-invariant and non-decreasing preferences⁵ $V(\cdot)$. Denote by X and Y two random variables. We say that $V(\cdot)$ is *non-decreasing*, if $X \leq Y$ almost surely implies that $V(X) \leq V(Y)$. We say that $V(\cdot)$ is *law-invariant* if $X \sim Y$ implies that $V(X) = V(Y)$. Our first result shows that these properties are equivalent to preferences

⁵This assumption is present in most traditional decision theories including expected utility theory (von Neumann and Morgenstern (1947)), Yaari's dual theory (Yaari (1987)), the cumulative prospect theory (Tversky and Kahneman (1992)) and rank dependent utility theory (Quiggin (1993)).

$V(\cdot)$ that respect first-order stochastic dominance (FSD). We denote by $X \prec_{fsd} Y$ when Y is (first-order) stochastically larger than X . This means that for all $x \in \mathbb{R}$, $F_X(x) \geq F_Y(x)$ where F_X and F_Y respectively denote the cumulative distribution functions (cdfs) of X and Y .

Theorem 1. *Preferences $V(\cdot)$ are non-decreasing and law-invariant if and only if $V(\cdot)$ satisfies first-order stochastic dominance.*

Proof. Assume that $V(\cdot)$ is non-decreasing and law-invariant and consider X and Y with respective distributions F_X and F_Y . Assume that $X \prec_{fsd} Y$. One has, for all $x \in (0, 1)$, $F_X^{-1}(x) \leq F_Y^{-1}(x)$. Let U be a uniform over $(0, 1)$. Hence, $F_X^{-1}(U) \leq F_Y^{-1}(U)$ a.s. Moreover $X \sim F_X^{-1}(U)$ and $Y \sim F_Y^{-1}(U)$ so that $V(X) = V(F_X^{-1}(U)) \leq V(F_Y^{-1}(U)) = V(Y)$. Thus $V(\cdot)$ satisfies FSD. Reciprocally, assume that $V(\cdot)$ satisfies FSD. If $X \sim Y$, then $X \prec_{fsd} Y$ and $Y \prec_{fsd} X$ which implies $V(X) = V(Y)$ and thus law invariance. Clearly, if $X \leq Y$ a.s. then $X \prec_{fsd} Y$ and $V(X) \leq V(Y)$. \square

An agent with preferences $V(\cdot)$ finds her optimal terminal consumption X_T^* by solving the optimization problem,

$$\max_{X_T \mid E[\xi_T X_T] = \omega_0} V(X_T). \quad (1)$$

An optimal strategy X_T^* (if it exists) has a particular distribution F . Intuitively speaking, since $V(\cdot)$ is non-decreasing and law-invariant, then among all strategies which are distributed with F , the optimum X_T^* is the cheapest possible one. This observation is made precise in the following theorem.

Theorem 2 (Cost-efficiency). *Assume that an optimum X_T^* of (1) exists and its cdf by F . Then, X_T^* is the cheapest way (also called cost-efficient) to achieve the distribution F at the investment horizon T , i.e. X_T^* also solves the following problem*

$$\min_{X_T \mid X_T \sim F} E[\xi_T X_T]. \quad (2)$$

Furthermore, for any given cdf F , the solution X_T^ to the dual investment problem (2) is unique almost surely (a.s.) and writes as $X_T^* = F^{-1}(1 - F_{\xi_T}(\xi_T))$. Thus payoffs are cost-efficient if and only if they are non-increasing in the state price ξ_T .*

We omit the proof as it is proved in Bernard, Boyle and Vanduffel (2011). The first part of the Theorem corresponds to their Proposition 5. The second

part corresponds to their Proposition 2 and Corollary 2. It is also closely related to results which first appeared in Dybvig (1988a), (1988b).

Theorem 3 provides us with a dual approach for portfolio optimization. Usually, one resorts to a value function $V(\cdot)$ to model preferences and then finds the optimal consumption by solving Problem (1). In the dual approach, one specifies a desired distribution F of terminal wealth up-front⁶ and determines the cheapest strategy that is distributed with F .

From Theorem 2, one obtains the following corollary immediately.

Corollary 1. *Assume that an optimum X_T^* of (1) exists and denote by F its cdf. Then X_T^* is almost surely non-decreasing in ξ_T and given as $X_T^* = F^{-1}(1 - F_{\xi_T}(\xi_T))$.*

In general, there may be more than one solution to (1). However, two different solutions must have different cdfs because the cost-efficient payoff generating a given distribution (obtained for a fixed budget) is unique. In the context of EUT, $V(X_T) = E(U(X_T))$ for some utility function $U(\cdot)$. When $U(\cdot)$ is not concave, a standard approach of solving Problem (1) is to introduce the concave envelope of $U(\cdot)$, denoted by $U_C(\cdot)$, which is the smallest concave function larger than or equal to $U(\cdot)$. Reichlin (2012) shows that under some technical assumptions, the maximizer for $U_C(\cdot)$ is also the maximizer for $U(\cdot)$. However, this maximizer is only unique under certain cases (see Lemma 5.9 of Reichlin (2012)).

In the next section, we reconcile different decision theories by showing that an optimal portfolio in any behavioral theory which respects first-order stochastic dominance can be obtained as an optimal portfolio for a risk averse expected utility maximizer.

3 Explaining Distributions through Expected Utility Theory

In the first part of this section, we show how the traditional expected utility setting with a strictly increasing and strictly concave utility function on an interval can be used to explain F when it is strictly increasing and continuous

⁶As aforementioned, it is presumably easier for many investors to describe a target terminal wealth distribution F than the value function $V(\cdot)$ governing their investment decision (see for instance Goldstein, Johnson and Sharpe (2008)).

on this interval. The second part shows that more generally, any distribution of optimal final wealth can be explained using a “generalized” utility function defined over \mathbb{R} . Since a solution X_T^* to the optimization problem (1) is completely characterized by its distribution F (Corollary 1), it follows that X_T^* is also optimal for some expected utility maximizer.

3.1 Standard Expected Utility Maximization

The following lemma finds the optimal payoff for an expected utility maximizer with a strictly increasing and strictly concave utility function. It is proved by Cox and Huang (1989) when Inada’s conditions are satisfied. We present the result in a slightly more general setting that is going to be useful in what follows. In particular, we show how the path-wise optimization technique, which we use repeatedly throughout the paper, can be used to solve the standard expected utility maximization problem (3).

Lemma 1. *Consider a utility function $U : (a, b) \rightarrow \mathbb{R}$, continuously differentiable and strictly increasing on $(a, b) \subset \mathbb{R}$ where $a, b \in \overline{\mathbb{R}}$ (with $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$). Furthermore, assume⁷ that U' is strictly decreasing on (a, b) (so that the buyer is risk averse), $U(c) = 0$ for some $c \in (a, b)$, $U'(a^+) := \lim_{x \searrow a} U'(x) = +\infty$ and $U'(b^-) := \lim_{x \nearrow b} U'(x) = 0$. The optimal solution X_T^* to the following portfolio optimization problem*

$$X_T \mid \max_{E[\xi_T X_T] = \omega_0} E[U(X_T)] \quad (3)$$

exists and is given by

$$X_T^* := [U']^{-1}(\lambda^* \xi_T)$$

where $\lambda^* > 0$ is such that $E[\xi_T [U']^{-1}(\lambda^* \xi_T)] = \omega_0$. Furthermore, X_T^* has a continuous distribution F which is strictly increasing on (a, b) with $F(a^+) = 0$, $F(b^-) = 1$.

Proof. Given $\omega \in \Omega$, consider the following auxiliary problem

$$\max_{y \in (a, b)} \{U(y) - \lambda \xi_T(\omega) y\} \quad (4)$$

with $\lambda \in \mathbb{R}^+$. This is an optimization over the interval (a, b) of a concave function. Under the stated conditions the optimum y^* is at $U'(y) - \lambda \xi_T(\omega) =$

⁷Note that $a = c = 0$ and $b = +\infty$ correspond to Inada’s conditions.

0, which can be computed explicitly: $y^* = [U']^{-1}(\lambda \xi_T(\omega))$. For each $\omega \in \Omega$, define the random variable Y_λ^* by $Y_\lambda^*(\omega) = y^*$ so that

$$Y_\lambda^*(\omega) = [U']^{-1}(\lambda \xi_T(\omega)).$$

Choose $\lambda^* > 0$ such that $E[\xi_T Y_{\lambda^*}^*] = \omega_0$. The existence of λ^* is ensured by the conditions imposed on $U(\cdot)$ and by continuity of $\lambda \mapsto E[\xi_T Y_\lambda^*]$. For every final wealth X_T which satisfies the budget constraint of (3), we have by construction,

$$U(X_T(\omega)) - \lambda^* \xi_T(\omega) X_T(\omega) \leq U(Y_{\lambda^*}^*(\omega)) - \lambda^* \xi_T(\omega) Y_{\lambda^*}^*(\omega)$$

since $Y_{\lambda^*}^*(\omega)$ is the optimal solution to (4). Furthermore, assume that $E[\xi_T X_T] = \omega_0$ and take the expectation on both sides of the above inequality. As $E[\xi_T X_T] = E[\xi_T Y_{\lambda^*}^*] = \omega_0$, we obtain

$$E[U(X_T)] \leq E[U(Y_{\lambda^*}^*)]$$

which ends the proof that $Y_{\lambda^*}^*$ is optimal for Problem (3). Since $[U']^{-1}(\cdot)$ is decreasing it is clear that $Y_{\lambda^*}^*$ has a continuous and strictly increasing distribution on (a, b) with $F(a^+) = 0$ and $F(b^-) = 1$. \square

The following theorem gives for any continuous strictly increasing distribution of final wealth an explicit construction of the utility function that explains the investor's demand via the expected utility maximization framework.

Theorem 3 (Strictly increasing continuous distribution). *Consider a cdf F that is strictly increasing and continuous on $(a, b) \subset \mathbb{R}$ with $a, b \in \overline{\mathbb{R}}$ so that $F(a^+) = 0$, $F(b^-) = 1$. Assume that the cost of the unique cost-efficient payoff X_T^* solving (2) is finite and denote it by ω_0 . Then X_T^* is also the optimal solution of the following expected utility maximization problem*

$$\max_{X_T \mid E[\xi_T X_T] = \omega_0} E[U(X_T)], \quad (5)$$

where $U : (a, b) \rightarrow \mathbb{R}$ is a continuously differentiable, strictly concave and strictly increasing utility function with $a, b \in \overline{\mathbb{R}}$. We can express it explicitly as

$$U(x) = \int_c^x F_{\xi_T}^{-1}(1 - F(y)) dy \quad (6)$$

for some $c > a$.

Proof. Consider the utility function $U(\cdot)$ as defined in (6). It is continuously differentiable on (a, b) and for $z \in (a, b)$,

$$U'(z) := F_{\xi_T}^{-1}(1 - F(z)). \quad (7)$$

Then U is strictly increasing and U' is strictly decreasing on (a, b) . Moreover $U(c) = 0$, $U'(a^+) = +\infty$ and $U'(b^-) = 0$ because ξ_T is continuously distributed on $\mathbb{R}^+ \setminus \{0\}$ and F is a cdf.

Using Lemma 1, the optimal solution to (5) can be written as

$$X_T^* := [U']^{-1}(\lambda^* \xi_T), \quad (8)$$

where $\lambda^* > 0$ is chosen such that $E[\xi_T X_T^*] = \omega_0$. Such a λ^* exists because of the conditions that U satisfies (see also Lemma 1). We can now rewrite (8) with the expression for $U'[\cdot]$ given in (7). In particular $[U']^{-1}(y) = F^{-1}(1 - F_{\xi_T}(y))$. We then find that

$$X_T^* = F^{-1}(1 - F_{\xi_T}(\lambda^* \xi_T)).$$

Observe that for $\lambda^* = 1$, $X_T^* \sim F$. Moreover X_T^* is non-decreasing in ξ_T and thus cost-efficient (see Theorem 2). By assumption, ω_0 is the cost of the unique cost-efficient payoff with distribution F . Therefore, $\lambda^* = 1$ ensures that $E[\xi_T X_T^*] = \omega_0$ and $X_T^* = [U']^{-1}(\xi_T)$ is thus the solution to the maximum expected utility problem (5). \square

If an expected utility maximizer prefers a particular payoff with cdf F , then her utility function is given by (6). One observes that this utility function is strictly increasing and strictly concave. It also seems to depend on the constant c . However, a utility function is defined up to a positive affine transformation and c has no effect on the expected utility maximization (5). It is also important to notice that this utility function (6) involves properties of the financial market at the horizon time T . Precisely, the utility function (6) involves the cdf of the pricing kernel ξ_T .

In the second part of this section, we generalize Theorem 3 to include any type of distribution (continuous, discrete and mixed distributions). Of course, any distribution F can always be approximated by a sequence of continuous strictly increasing distributions, F_n . Then, for each F_n , Theorem 3 allows us to obtain the corresponding strictly concave and strictly increasing utility function, U_n so that the optimal investment for an expected utility maximizer with utility function U_n is distributed with the cdf F_n . Therefore, Theorem 3 already explains *approximately* the demand for a discrete distribution.

However, the analysis of the general case is also of economic interest. He and Zhou (2011b) show that under some assumptions, optimal payoffs in Yaari's dual theory have a discrete distribution whereas in the case of cumulative prospect theory, the optimal final wealth has a mixed distribution. While these observations point to differences between the decision theories, we show that expected utility theory can also explain these optimal payoffs. Note also that in practice many financial products exhibit guarantees and thus have payoffs with a mixed distribution.

3.2 Generalized Expected Utility Maximization

In the previous section, a utility function is continuously differentiable, strictly concave and strictly increasing on an interval (a, b) with $a, b \in \overline{\mathbb{R}}$. This allows to explain the demand for continuous strictly increasing distributions. In this section, we consider "generalized" utility functions defined on the entire real line \mathbb{R} . This allows to explain the demand for *all* distributions.

Definition 1 (Generalized utility function). *A generalized utility function $\tilde{U} : \mathbb{R} \rightarrow \mathbb{R}$ is defined as*

$$\tilde{U}(x) := \begin{cases} U(x) & \text{for } x \in (a, b), \\ -\infty & \text{for } x < a, \\ U(a^+) & \text{for } x = a, \\ U(b^-) & \text{for } x \geq b, \end{cases}$$

where $U(x)$ is concave and strictly increasing and $(a, b) \subset \mathbb{R}$.

A generalized utility function does not need to be differentiable on (a, b) nor strictly concave. A generalized utility function \tilde{U} is said to be non-decreasing on \mathbb{R} and concave⁸ if it is strictly increasing and concave on (a, b) .

The following theorem shows that a continuously distributed optimal terminal wealth in a decision theory which satisfies FSD can always be explained by expected utility maximization using a generalized utility as defined in Definition 1.

Theorem 4 (Continuous Distribution). *Let F be a continuous distribution on (a, b) . Let X_T^* be the solution to (2) for this cdf F . Assume its cost, ω_0 ,*

⁸Concavity is a concept for real-valued functions only and here \tilde{U} takes values in $\overline{\mathbb{R}}$.

is finite. Let $c > a$. Then X_T^* is also an optimal solution to the following expected utility maximization problem

$$\max_{X_T \mid E[\xi_T X_T] = \omega_0} E[\tilde{U}(X_T)] \quad (9)$$

where $\tilde{U} : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is a generalized utility function defined as

$$\tilde{U}(x) = U(x) := \int_c^x F_{\xi_T}^{-1}(1 - F(y)) dy, \quad (10)$$

for all $a < x < b$, $\tilde{U}(x) = -\infty$ for $x < a$, $\tilde{U}(a) = U(a^+)$ and $\tilde{U}(x) = U(b^-)$ for $b \leq x$.

Proof. Take \tilde{U} as defined in (10). Given $\omega \in \Omega$, consider the following auxiliary problem

$$\max_{y \in \mathbb{R}} \left\{ \tilde{U}(y) - \lambda \xi_T(\omega) y \right\} \quad (11)$$

with $\lambda \in \mathbb{R}^+$. Note that \tilde{U} is differentiable on (a, b) and $\tilde{U}'(x) = F_{\xi_T}^{-1}(1 - F(x))$. However, F is not strictly increasing and thus to invert \tilde{U}' , we consider the following pseudo-inverse, defined for all $y \geq 0$,

$$\left[\tilde{U}' \right]^{-1}(y) = \inf \left\{ x \in (a, b) \mid \tilde{U}'(x) \geq y \right\}. \quad (12)$$

Since \tilde{U} is differentiable and concave on (a, b) , one has that

$$Y_\lambda^*(\omega) := \max \left(a, \min \left[\tilde{U}' \right]^{-1}(\lambda \xi_T(\omega)), b \right),$$

is an optimum of

$$\max_{y \in [a, b]} \left\{ \tilde{U}(y) - \lambda \xi_T(\omega) y \right\}.$$

Furthermore, for $\lambda \geq 0$, it is clear that

$$\forall z \geq b, \quad \tilde{U}(z) - \lambda \xi_T(\omega) z \leq \tilde{U}(b) - \lambda \xi_T(\omega) b,$$

because $\tilde{U}(b) = \tilde{U}(z)$, whereas $\forall z < a$, $\tilde{U}(z) - \lambda \xi_T(\omega) z = -\infty$. Hence, $Y_\lambda^*(\omega)$ solves (11). Using (10) and (12) we find that

$$Y_\lambda^*(\omega) = \max \left(a, \min \left(b, F^{-1}(1 - F_{\xi_T}(\lambda \xi_T)) \right) \right) = F^{-1}(1 - F_{\xi_T}(\lambda \xi_T)),$$

where the last equality holds because $F^{-1}(\cdot)$ takes values in $[a, b]$. If there exists $\lambda > 0$ such that the path-wise optimum Y_λ^* satisfies the budget constraint $E[\xi_T Y_\lambda^*] = \omega_0$, then Y_λ^* solves Problem (9). Note that for $\lambda = 1$, Y_1^* is the unique cost-efficient payoff distributed with F and solving (2). By assumption, its cost is ω_0 . Thus, Y_1^* solves (9). \square

In the following theorem we find a generalized utility function to explain the demand for any given discrete distribution of final wealth.

Theorem 5 (Discrete Distribution). *Let F be a discrete distribution and let $\mathcal{S} = \{x_i, i \in I\}$ be the set of its mass points⁹ with infimum m and supremum M . Let X_T^* be the solution to (2) for cdf F . Assume its cost, ω_0 , is finite. Let $c > m$. Define the generalized utility function $\tilde{U} : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ as,*

$$\tilde{U}(x) := \begin{cases} U(x) = \int_c^x F_{\xi_T}^{-1}(1 - F(y))dy, & \text{for } x \in (m, M), \\ -\infty & \text{for } x < m, \\ U(m^+) & \text{for } x = m, \\ U(M^-) & \text{for } x \geq M, \end{cases} \quad (13)$$

with the convention that $U(m^+) = U(M^-) = 0$ when $m = M$. Then X_T^* is also an optimal solution to the following expected utility maximization problem,

$$\max_{X_T \mid E[\xi_T X_T] = \omega_0} E[\tilde{U}(X_T)]. \quad (14)$$

Proof. Consider the utility function \tilde{U} as defined in (13). As F is a discrete distribution, for $x \in (m, M) \setminus \mathcal{S}$, \tilde{U} is differentiable and $\tilde{U}'(x) = F_{\xi_T}^{-1}(1 - F(x))$. For $x_i \in (m, M) \cap \mathcal{S}$, we take the right derivative of \tilde{U} so that $\tilde{U}'(x_i) = F_{\xi_T}^{-1}(1 - F(x_i))$. Thus \tilde{U}' is a non-increasing step function on (m, M) . For all $i \in I$, we define $a_i^{(r)}$ and $a_i^{(\ell)}$ as

$$\begin{aligned} a_i^{(r)} &:= \tilde{U}'(x_i) = F_{\xi_T}^{-1}(1 - F(x_i)), \\ a_i^{(\ell)} &:= \tilde{U}'(x_i^-) = F_{\xi_T}^{-1}(1 - F(x_i^-)) = F_{\xi_T}^{-1}(1 - F(x_{i-1})), \text{ if } x_i > m, \\ a_i^{(\ell)} &:= +\infty, \text{ if } x_i = m. \end{aligned}$$

It follows that $a_i^{(\ell)} > a_i^{(r)} = a_{i+1}^{(\ell)}$; thus, $\tilde{U}(x)$ is strictly increasing and concave on (m, M) . Hence it is a generalized utility function. To maximize

⁹Without loss of generality I is an index set such that $x_i < x_j$ if and only if $i < j$.

$E[\tilde{U}(X_T)]$, we proceed by pathwise optimization. Given $\omega \in \Omega$, consider the following auxiliary problem

$$\max_{y \in \mathbb{R}} \left\{ \tilde{U}(y) - \lambda \xi_T(\omega) y \right\} \quad (15)$$

with $\lambda \in \mathbb{R}^+$. Denote by $Y_\lambda^*(\omega)$ the optimum of (15) when it exists. As $a_i^{(\ell)} = +\infty$ if $x_i = m$ and $a_i^{(r)} = 0$ for $x_i = M$, there exists $i \in I$ such that $a_i^{(r)} \leq \lambda \xi_T(\omega) < a_i^{(\ell)}$. Let us show that $Y_\lambda^*(\omega) = x_i$. To do so, we compare $U(x_i) - \lambda \xi_T(\omega) x_i$ and $U(x) - \lambda \xi_T(\omega) x$ for $x \in \mathbb{R}$. For all $x < m$,

$$\tilde{U}(x) - \lambda \xi_T(\omega) x = -\infty < \tilde{U}(x_i) - \lambda \xi_T(\omega) x_i.$$

For all $x \geq M$, $\tilde{U}(x) = \tilde{U}(M)$, thus

$$\tilde{U}(x) - \lambda \xi_T(\omega) x \leq \tilde{U}(M) - \lambda \xi_T(\omega) M.$$

Thus the optimum to (15), if it exists, belongs to $[m, M]$. Assume $x \in (m, M)$, then $\tilde{U}(x) = U(x)$ and we discuss two cases: $x < x_i$ and $x > x_i$. When $x < x_i$,

$$\begin{aligned} \tilde{U}(x_i) - \lambda \xi_T(\omega) x_i - (\tilde{U}(x) - \lambda \xi_T(\omega) x) &= \left[\frac{\tilde{U}(x_i) - \tilde{U}(x)}{x_i - x} - \lambda \xi_T(\omega) \right] (x_i - x) \\ &\geq (a_i^{(\ell)} - \lambda \xi_T(\omega))(x_i - x) > 0, \end{aligned}$$

where the first inequality follows from the concavity of $U(x)$ on (m, M) . When $x > x_i$, using the concavity of $U(\cdot)$ again on (m, M) , we find that

$$\begin{aligned} \tilde{U}(x_i) - \lambda \xi_T(\omega) x_i - (\tilde{U}(x) - \lambda \xi_T(\omega) x) &= \left[-\frac{\tilde{U}(x) - \tilde{U}(x_i)}{x - x_i} + \lambda \xi_T(\omega) \right] (x - x_i) \\ &\geq (-a_i^{(r)} + \lambda \xi_T(\omega))(x - x_i) \geq 0. \end{aligned}$$

For all $x \in (m, M)$, we have proved that $\tilde{U}(x_i) - \lambda \xi_T(\omega) x_i - (\tilde{U}(x) - \lambda \xi_T(\omega) x) \geq 0$. When $m < M$, $\tilde{U}(\cdot)$ is a continuous function of x on $[m, M]$, we have that for all $x \in [m, M]$,

$$\tilde{U}(x_i) - \lambda \xi_T(\omega) x_i - (\tilde{U}(x) - \lambda \xi_T(\omega) x) \geq 0.$$

This also holds when $x_i = m = M$ directly.

We have proved that for each $\omega \in \Omega$, and $\lambda \geq 0$,

$$Y_\lambda^*(\omega) = \sum_{i \in I} x_i \mathbb{1}_{a_i^{(r)} \leq \lambda \xi_T(\omega) < a_i^{(\ell)}}$$

solves the auxiliary problem (15). If there exists $\lambda \geq 0$ such that the path-wise optimum Y_λ^* satisfies the budget constraint $E[\xi_T Y_\lambda^*] = \omega_0$, then Y_λ^* solves Problem (14). Observe that for $\lambda = 1$,

$$P(Y_1^* = x_i) = P\left[a_i^{(r)} \leq \xi_T < a_i^{(\ell)}\right] = F_{\xi_T}\left(a_i^{(\ell)}\right) - F_{\xi_T}\left(a_i^{(r)}\right) = F(x_i) - F(x_i^-).$$

This shows that the distribution of Y_1^* is exactly the cdf F . Moreover, it is clear that Y_1^* is non-increasing in ξ_T and therefore is cost-efficient (see Theorem 2). By assumption, ω_0 is the cost of the cost-efficient payoff with distribution F . Therefore $E[\xi_T Y_1^*] = \omega_0$ and Y_1^* is thus a solution to the maximum expected utility problem (14) with respect to the stated $\tilde{U}(x)$. \square

Consider a given mixed distribution function $F(\cdot)$. Since it is right continuous and non-decreasing, it has at most countably many atoms. It follows that F can always be expressed as $F = pF^d + (1-p)F^c$, where $0 \leq p \leq 1$, and F^d, F^c are discrete and continuous distributions respectively (Jordan's decomposition result; see p.138 in Feller (1971), Volume 2).

Theorem 6 (Mixed Distribution). *Let F be a distribution with $F = pF^d + (1-p)F^c$, $0 < p < 1$ and F^d (resp. F^c) is a discrete (resp. continuous) distribution. Let $\mathcal{S} = \{x_i, i \in I\}$ be the set of its mass points, with infimum \underline{m} and supremum \overline{M} . Denote $\underline{m} = \inf\{x \mid F(x^-) > 0\}$ and $\overline{M} = \sup\{x \mid F(x^-) < 1\}$. Let X_T^* the optimal solution to (2) for cdf F . Denote by ω_0 its cost and assume it is finite. Let $c > \underline{m}$. Then X_T^* is also the optimal solution to the following expected utility maximization problem*

$$\max_{X_T \mid E[\xi_T X_T] = \omega_0} E\left[\tilde{U}(X_T)\right] \quad (16)$$

where $\tilde{U} : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is a generalized utility function defined as

$$\tilde{U}(x) := U(x) = \int_c^x F_{\xi_T}^{-1}(1 - (pF^d(y) + (1-p)F^c(y)))dy, \quad (17)$$

for all $\underline{m} < x < \overline{M}$, $\tilde{U}(x) = -\infty$ for $x < \underline{m}$, $\tilde{U}(\underline{m}) = U(\underline{m}^+)$ and $\tilde{U}(x) = U(\overline{M}^-)$ for $\overline{M} \leq x$.

Proof. Consider the generalized utility defined in (17). For all $x \in (\underline{m}, \overline{M})$, \tilde{U} is either differentiable at x (if $x \notin \mathcal{S}$) or has a right derivative at x (if $x \in \mathcal{S}$). In all cases, we write $\tilde{U}'(x)$ to refer to this right derivative. For all

$i \in I$, we define $a_i^{(r)}$ and $a_i^{(\ell)}$ as

$$\begin{aligned} a_i^{(r)} &:= \tilde{U}'(x_i) = F_{\xi_T}^{-1}(1 - pF^d(x_i) - (1-p)F^c(x_i)), \\ a_i^{(\ell)} &:= \tilde{U}'(x_i^-) = F_{\xi_T}^{-1}(1 - pF^d(x_i^-) - (1-p)F^c(x_i)), \\ &= F_{\xi_T}^{-1}(1 - pF^d(x_{i-1}) - (1-p)F^c(x_i)). \end{aligned}$$

where $a_i^{(\ell)} = \infty$ if $x_i = m = \underline{m}$, $a_i^{(\ell)} = F_{\xi_T}^{-1}(1 - (1-p)F^c(x_i))$ if $x_i = m > \underline{m}$. Note that $a_i^{(\ell)} > a_i^{(r)}$ and that the utility function $\tilde{U}(x)$ as defined in (17) is concave on $(\underline{m}, \overline{M})$. To maximize $E[\tilde{U}(X_T)]$, we proceed by path-wise optimization. Given $\omega \in \Omega$ and $\lambda \in \mathbb{R}^+$, consider the following auxiliary problem

$$\max_{y \in \mathbb{R}} \left\{ \tilde{U}(y) - \lambda \xi_T(\omega)y \right\}. \quad (18)$$

Note similarly as in the proofs of Theorems 4 and 5 that the optimum, if it exists, must be in $[\underline{m}, \overline{M}]$.

Observe that for $i \in I$ and $i+1 \in I$, $a_{i+1}^{(r)} < a_{i+1}^{(\ell)} \leq a_i^{(r)} < a_i^{(\ell)}$ and $a_i^{(r)}$ may be strictly larger than $a_{i+1}^{(\ell)}$ so that we need to consider two cases.

Case 1: There exists $i \in I$ such that $a_i^{(r)} \leq \lambda \xi_T(\omega) < a_i^{(\ell)}$. This part of the proof is similar to the proof of Theorem 5. We can show that $Y_\lambda^*(\omega) = x_i$ is the optimal solution to the auxiliary problem (18). To prove it, we observe that when $\underline{m} < x < x_i$, by concavity¹⁰ of $\tilde{U}(x)$,

$$\begin{aligned} \tilde{U}(x_i) - \lambda \xi_T(\omega)x_i - \tilde{U}(x) + \lambda \xi_T(\omega)x &= \left(\frac{\tilde{U}(x_i) - \tilde{U}(x)}{x_i - x} - \lambda \xi_T(\omega) \right) (x_i - x) \\ &\geq (a_i^{(\ell)} - \lambda \xi_T(\omega))(x_i - x_j) > 0, \end{aligned}$$

When $\overline{M} > x > x_i$, using the concavity of $\tilde{U}(x)$ again,

$$\begin{aligned} \tilde{U}(x_i) - \lambda \xi_T(\omega)x_i - \tilde{U}(x) + \lambda \xi_T(\omega)x &= \left(-\frac{\tilde{U}(x) - \tilde{U}(x_i)}{x - x_i} + \lambda \xi_T(\omega) \right) (x - x_i) \\ &\geq (-a_i^{(r)} + \lambda \xi_T(\omega))(x - x_i) \geq 0. \end{aligned}$$

For all $x \in (m, M)$ we have proved that $\tilde{U}(x_i) - \lambda \xi_T(\omega)x_i - (\tilde{U}(x) - \lambda \xi_T(\omega)x) \geq 0$. When $m < M$, $\tilde{U}(\cdot)$ is a continuous function of x on $[m, M]$, we have that for all $x \in [m, M]$,

$$\tilde{U}(x_i) - \lambda \xi_T(\omega)x_i - (\tilde{U}(x) - \lambda \xi_T(\omega)x) \geq 0.$$

¹⁰This is the concavity over $(\underline{m}, \overline{M})$ as explained in the definition of the generalized utility function in Definition 1.

This also holds when $x_i = m = M$ directly.

Case 2: There exists no $i \in I$ such that $a_i^{(r)} \leq \lambda \xi_T(\omega) < a_i^{(\ell)}$.

In this case, $\lambda \xi_T(\omega)$ belongs to the image of \tilde{U}' and $x^* := [\tilde{U}']^{-1}(\lambda \xi_T(\omega))$ is well-defined. Let us prove that $Y_\lambda^*(\omega) = x^*$ is the optimal solution to the auxiliary problem (18). We discuss two cases. When $\underline{m} < x < x^*$, then,

$$\begin{aligned} \tilde{U}(x^*) - \lambda \xi_T(\omega)x^* - (\tilde{U}(x) - \lambda \xi_T(\omega)x) &= \left[\frac{\tilde{U}(x^*) - \tilde{U}(x)}{x^* - x} - \lambda \xi_T(\omega) \right] (x^* - x) \\ &> (\lambda \xi_T(\omega) - \lambda \xi_T(\omega))(x^* - x) = 0, \end{aligned}$$

where we use the concavity of $\tilde{U}(x)$ (precisely the fact that the slope $\frac{\tilde{U}(x^*) - \tilde{U}(x)}{x^* - x}$ is larger than $\tilde{U}'(x^*)$). When $\overline{M} > x > x^*$, then using the concavity of $\tilde{U}(x)$, we have that

$$\begin{aligned} \tilde{U}(x^*) - \lambda \xi_T(\omega)x^* - (\tilde{U}(x) - \lambda \xi_T(\omega)x) &= \left[-\frac{\tilde{U}(x) - \tilde{U}(x^*)}{x - x^*} + \lambda \xi_T(\omega) \right] (x - x^*) \\ &\geq (-\lambda \xi_T(\omega) + \lambda \xi_T(\omega))(x - x_i) = 0. \end{aligned}$$

For all $x \in (m, M)$ we have proved that $\tilde{U}(x^*) - \lambda \xi_T(\omega)x^* - (\tilde{U}(x) - \lambda \xi_T(\omega)x) \geq 0$. When $m < M$, $\tilde{U}(\cdot)$ is a continuous function of x on $[m, M]$, we have that for all $x \in [m, M]$,

$$\tilde{U}(x^*) - \lambda \xi_T(\omega)x^* - (\tilde{U}(x) - \lambda \xi_T(\omega)x) \geq 0.$$

This also holds when $x_i = m = M$ directly.

For all $\omega \in \Omega$ we are thus able to find an optimum to (18). We now need to determine λ such that $E[\xi_T Y_\lambda^*] = \omega_0$ to ensure that Y_λ^* solves the original problem (16).

$$Y_\lambda^*(\omega) = \sum_{i \in I} x_i \mathbb{1}_{a_i^{(r)} \leq \lambda \xi_T(\omega) < a_i^{(\ell)}} + [\tilde{U}']^{-1}(\lambda \xi_T(\omega)) \left(1 - \sum_{i \in I} \mathbb{1}_{a_i^{(r)} \leq \lambda \xi_T(\omega) < a_i^{(\ell)}} \right).$$

It is clear from this formula that the first part of the above expression corresponds to the discrete part and the second part is continuously distributed. Firstly, note that

$$P(Y_1^* = x_i) = P\left[a_i^{(r)} \leq \xi_T < a_i^{(\ell)}\right] = F_{\xi_T}\left(a_i^{(\ell)}\right) - F_{\xi_T}\left(a_i^{(r)}\right) = F(x_i) - F(x_i^-),$$

so that Y_1^* has the mass points with the right probabilities. Furthermore, we have that

$$P(Y_1^* \leq x) = F(x)$$

for all $x \in \mathbb{R}$ so that Y_1^* is distributed with F . We observe also that it is non-increasing in ξ_T therefore is cost-efficient (See Theorem 2). By assumption, ω_0 is the cost of the cost-efficient payoff with cdf F . Therefore, by uniqueness of the cost-efficient payoffs with cdf F , $E[\xi_T Y_1^*] = \omega_0$. Hence, Y_1^* is a solution to the maximum expected utility problem with respect to the stated $\tilde{U}(x)$. \square

4 From Distributions to Utility Functions

In this section, we use the results of the previous section to derive utility functions that explain the observed demand of agents in frameworks that satisfy first-order stochastic dominance. Let us start with a simple example showing how one can recover the popular CRRA utility function from a lognormally distributed final wealth. This example is particularly useful when explaining the optimal demand for a retail investor who chooses an equity-linked structured product with capital guarantee. The last example concerns a reconciliation of Yaari's dual theory with expected utility theory. Precisely, we exhibit the non-decreasing concave utility function such that the optimal strategy in Yaari's (1987) theory is also obtained in an expected utility maximization framework.

For the ease of exposition, we restrict ourselves to the Black-Scholes model¹¹. In this case, the state price ξ_T is unique and can be expressed explicitly in terms of the stock price S_T as follows

$$\xi_T = \alpha \left(\frac{S_T}{S_0} \right)^{-\beta}, \quad (19)$$

where $\frac{S_T}{S_0} \sim \mathcal{LN} \left(\left(\mu - \frac{\sigma^2}{2} \right), \sigma^2 T \right)$, $\alpha = \exp \left(\frac{\theta}{\sigma} \left(\mu - \frac{\sigma^2}{2} \right) T - \left(r + \frac{\theta^2}{2} \right) T \right)$, $\beta = \frac{\theta}{\sigma}$ and $\theta = \frac{\mu - r}{\sigma}$. Formula (19) is well-known and can be found for example in Section 3.3 of Bernard *et al.* (2011). It follows that

$$\xi_T \sim \mathcal{LN} \left(-rT - \frac{\theta^2 T}{2}, \theta^2 T \right). \quad (20)$$

Assume first that consumption is restricted on $(0, \infty)$ and that the investor wants to achieve a lognormal distribution $\mathcal{LN}(M, \Sigma^2)$ at maturity T

¹¹All developments can be done in the general market setting given in Section 2 but closed-form solutions are more complicated or unavailable.

of her investment. The desired cdf is $F(x) = \Phi\left(\frac{\ln x - M}{\Sigma}\right)$, and from (19) it follows that $F_{\xi_T}^{-1}(y) = \exp\left\{\Phi^{-1}(y)\theta\sqrt{T} - rT - \frac{\theta^2 T}{2}\right\}$. Applying Theorem 3, the utility function explaining this distribution writes as

$$U(x) = \begin{cases} a \frac{x^{1-\frac{\theta\sqrt{T}}{\Sigma}}}{1-\frac{\theta\sqrt{T}}{\Sigma}} & \frac{\theta\sqrt{T}}{\Sigma} \neq 1 \\ a \log(x) & \frac{\theta\sqrt{T}}{\Sigma} = 1 \end{cases}, \quad (21)$$

where $a = \exp(\frac{M\theta\sqrt{T}}{\Sigma} - rT - \frac{\theta^2 T}{2})$. This is a CRRA utility function with relative risk aversion $\frac{\theta\sqrt{T}}{\Sigma}$. A more thorough treatment of risk aversion comes in Section 5.

4.1 Explaining the Demand for Capital Guarantee Products

Many structured products include a capital guarantee and have a payoff of the form,

$$Y_T = \max(G, S_T),$$

where S_T is the stock price. It has a lognormal distribution $S_T \sim \mathcal{LN}(M, \Sigma^2)$ where $M = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T$ and $\Sigma^2 = \sigma^2 T$, so that the cdf for Y_T is given by

$$F_{Y_T}(y) = \begin{cases} 0 & y < G \\ \Phi\left(\frac{\ln y - M}{\Sigma}\right) & y \geq G \end{cases}. \quad (22)$$

Since Y_T has a mixed distribution, we can apply Theorem 6 to derive the corresponding utility function. Let $p := \Phi\left(\frac{\ln G - M}{\Sigma}\right)$ and define the following discrete and continuous cdfs

$$F_{Y_T}^D(y) = \begin{cases} 0 & y < G \\ 1 & y \geq G \end{cases}, \quad (23)$$

$$F_{Y_T}^C(y) = \begin{cases} 0 & y < G \\ \frac{\Phi\left(\frac{\ln y - M}{\Sigma}\right) - p}{1 - p} & y \geq G \end{cases}. \quad (24)$$

Then, we can see that $F_{Y_T}(y) = pF_{Y_T}^D(y) + (1 - p)F_{Y_T}^C(y)$. The set of mass points is a singleton $\mathcal{S} = \{G\}$ so $m = M = G$. Moreover, $\underline{m} = G$ and $\overline{M} = \infty$. When $x < G$, $\tilde{U}(x) = -\infty$. The utility function is then given by

$$\tilde{U}(x) = \begin{cases} -\infty & x < G, \\ a \frac{x^{1-\frac{\theta\sqrt{T}}{\Sigma}} - G^{1-\frac{\theta\sqrt{T}}{\Sigma}}}{1-\frac{\theta\sqrt{T}}{\Sigma}} & x \geq G, \frac{\theta\sqrt{T}}{\Sigma} \neq 1, \\ a \log\left(\frac{x}{G}\right) & x \geq G, \frac{\theta\sqrt{T}}{\Sigma} = 1, \end{cases} \quad (25)$$

with $a = \exp(\frac{M\theta\sqrt{T}}{\Sigma} - rT - \frac{\theta^2 T}{2})$. The mass point is explained by a utility which is infinitely negative for any level of wealth below the guaranteed level. The CRRA utility above this guaranteed level ensures the optimality of a lognormal distribution above the guarantee, as aforementioned.

4.2 Yaari's Dual Theory of Choice Model

Optimal portfolio selection under Yaari's (1987) dual theory involves maximizing the expected value of terminal payoff under a distorted probability function. Precisely, under Yaari's dual theory of choice, decision makers evaluate the "utility" of their non-negative final wealth X_T (with cdf F) by calculating its distorted expectation $\mathbb{H}_w[X_T]$

$$\mathbb{H}_w[X_T] = \int_0^\infty w(1 - F(x)) dx, \quad (26)$$

where the (distortion) function $w : [0, 1] \rightarrow [0, 1]$ is non-decreasing with $w(0) = 0$ and $w(1) = 1$.

The investor's initial endowment is $\omega_0 \geq 0$. He and Zhou (2011b) find the optimal payoff when the distortion function is given by $w(z) = z^\gamma$ where $\gamma > 1$. They show that the following function

$$\begin{aligned} & \left[\Phi \left(\frac{\ln x + rT + \frac{\theta^2 T}{2}}{\theta\sqrt{T}} \right) \right]^{\gamma-1} \\ & \times \left[x \Phi \left(\frac{\ln x + rT + \frac{\theta^2 T}{2}}{\theta\sqrt{T}} \right) - \gamma e^{-rT} \Phi \left(\frac{\ln x + rT + \frac{\theta^2 T}{2}}{\theta\sqrt{T}} \right) \right] \end{aligned}$$

has a unique root c on $(1, \gamma e^{-rT})$ and that the optimal payoff is equal to

$$X_T^* = b \mathbb{1}_{\xi_T \leq c}, \quad (27)$$

where $b = \omega_0 e^{rT} \Phi \left(\frac{\ln c + rT - \frac{\theta^2 T}{2}}{\theta\sqrt{T}} \right) > 0$ is chosen such that the budget constraint $E[\xi_T X_T^*] = \omega_0$ is fulfilled. The corresponding cdf is

$$F(x) = \begin{cases} \Phi \left(\frac{-rT - \frac{\theta^2 T}{2} - \ln c}{\theta\sqrt{T}} \right) & 0 \leq x < b \\ 1 & x \geq b \end{cases}. \quad (28)$$

In this case, the distribution is discrete, so we apply Theorem 5. With the notation introduced in Theorem 5, the set of mass points is $\mathcal{S} = \{0, b\}$, so $m = 0$ and $M = b$. We find that the utility function is given by

$$\tilde{U}(x) = \begin{cases} -\infty & x < 0, \\ c(x - c) & 0 \leq x \leq b, \\ c(b - c) & x > b. \end{cases} \quad (29)$$

The utility function such that the optimal investment in the expected utility setting is similar to the optimum in Yaari's framework (1987) is simply linear up to a maximum $c(b - c)$ and then constant.

5 From Distributions to Risk Aversion

In this section, we first derive expressions for the Arrow-Pratt measures for risk aversion as a function of the distributional properties of the terminal wealth and the financial market (we thus implicitly assume that agents act as expected utility optimizers). Next, we show that decreasing absolute risk aversion (DARA) is equivalent to terminal wealth exhibiting more spread than the market variable $H_T := -\log(\xi_T)$. In a Black-Scholes setting, agents thus have DARA preferences if and only if they show a demand for “fatter than normal” tailed distributions. Our characterization for DARA also allows to construct an empirical test for DARA preferences based on observed investment behavior.

In what follows, we assume that H_T is absolutely continuous (on \mathbb{R}) with distribution G and density g such that $g(x) > 0$ for all $x \in \mathbb{R}$. In other words, the density of ξ_T must exist and must be positive on $\mathbb{R}^+ \setminus \{0\}$.

5.1 Risk Aversion Coefficient

Recall that the Arrow-Pratt measures for absolute and relative risk aversion, $\mathcal{A}(x)$ resp. $\mathcal{R}(x)$ can be computed from a twice differentiable utility function U as $\mathcal{A}(x) = -\frac{U''(x)}{U'(x)}$ and $\mathcal{R}(x) = x\mathcal{A}(x)$.

Theorem 7 (Arrow-Pratt measures for risk aversion). *Consider an investor who wants an absolutely continuous cdf F for his terminal wealth (with corresponding density f). Consider a level of wealth x in the interior of the support of the distribution F . Then $x = F^{-1}(p)$ for some $0 < p < 1$. The*

Arrow-Pratt measures for respectively absolute and relative risk aversion at x are given as

$$\mathcal{A}(x) = \frac{f(F^{-1}(p))}{g(G^{-1}(p))}, \quad (30)$$

$$\mathcal{R}(x) = F^{-1}(p) \frac{f(F^{-1}(p))}{g(G^{-1}(p))}. \quad (31)$$

Proof. Using the expression for U obtained in (6), we can easily derive,

$$U'(x) = F_{\xi_T}^{-1}(1 - F(x)) = \exp(-G^{-1}(F(x)))$$

and

$$U''(x) = U'(x) \frac{-f(x)}{g(G^{-1}(F(x)))}.$$

The stated expression for the absolute risk aversion coefficient $\mathcal{A}(x)$ is then obtained, as $\mathcal{A}(x) = -\frac{U''(x)}{U'(x)}$. It is well-defined when $F(x) \in (0, 1)$ because $g(x) > 0$ for all $x \in \mathbb{R}$ and G^{-1} exists on $(0, 1)$. The expression for the relative risk aversion coefficient $\mathcal{R}(x)$ follows immediately. \square

Expression (30) shows that the coefficient for absolute risk aversion can be interpreted as a likelihood ratio and is directly linked to the financial market (through the cdf G of the opposite of the log pricing kernel).

5.2 Decreasing Absolute Risk Aversion

We provide precise characterizations of DARA in terms of distributional properties of the final wealth and the financial market. In what follows, we only consider distributions that are twice differentiable.

Theorem 8 (Distributional characterization of DARA). *Consider an investor who targets some absolutely continuous distribution F for his terminal wealth. Denote by f its density. The investor has (strictly) decreasing absolute risk aversion (DARA) if and only if,*

$$y \mapsto F^{-1}(G(y)) \text{ is strictly convex on } \mathbb{R}. \quad (32)$$

The investor has asymptotic DARA if and only if there exists $y^ \in \mathbb{R}$ such that $y \mapsto F^{-1}(G(y))$ is strictly convex on (y^*, ∞) .*

Proof. Denote by \mathbb{F} the domain where $\mathcal{A}(x)$ is well-defined. To assess the DARA property we study when $\ln(\mathcal{A}(x))$ is strictly decreasing for $x \in \mathbb{F}$. For all $x \in \mathbb{F}$, it is clear that $x \mapsto \ln(\mathcal{A}(x))$ is decreasing if and only if $p \mapsto \ln(\mathcal{A}(F^{-1}(p)))$ is decreasing on $(0, 1)$. Using expression (30) for $\mathcal{A}(x)$ for $x \in \mathbb{F}$ and differentiating with respect to p , one has that the derivative is negative if and only if for $p \in (0, 1)$

$$\frac{g'(G^{-1}(p))}{g^2(G^{-1}(p))} > \frac{f'(F^{-1}(p))}{f^2(F^{-1}(p))}. \quad (33)$$

Consider next the auxiliary function $y \mapsto F^{-1}(G(y))$ defined on \mathbb{R} . By twice differentiating and using the substitution $q = G(y)$, one finds that it is strictly convex if and only if for $q \in (0, 1)$,

$$\frac{g'(G^{-1}(q))}{g^2(G^{-1}(q))} > \frac{f'(F^{-1}(q))}{f^2(F^{-1}(q))}. \quad (34)$$

Since (33) and (34) are the same, this ends the proof. \square

The convexity of the function $F^{-1}(G(x))$ reflects that the target distribution F is “fatter tailed” than the distribution G . In the literature, it is said that F is larger than G in the sense of transform convex order (Shaked and Shantikumar (2007), p.214).

Theorem 8 extends recent results by Dybvig and Wang (2012) and Beiglöck, Muhle-Karbe and Temme (2012) in another direction. They show that if agent A has lower risk aversion than agent B , then agent A purchases a distribution that is larger than the other in the sense of second-order stochastic dominance. Here, we show that the risk aversion of an agent is decreasing in available wealth if and only if the agent purchases a payoff that is heavier tailed than the market variable H_T .

Theorem 9. *Consider an investor with optimal terminal wealth $W_T \sim F$. The investor has decreasing absolute risk aversion if and only if W_T is increasing and strictly convex in H_T .*

Proof. Assume the investor has the DARA property. Consider the variable $Y_T = F^{-1}(G(H_T))$ and observe that $Y_T \sim F$ (because $G(H_T) \sim \mathcal{U}(0, 1)$). One has that $k(x) := F^{-1}(G(x))$ is strictly convex, and since it is also increasing, we only need to prove that $Y_T = W_T$ almost surely. Y_T is non-increasing in ξ_T and thus cost-efficient. It is thus the almost surely unique cost-efficient payoff distributed with F . Since W_T is also cost-efficient and

distributed with F it follows that $Y_T = W_T$ almost surely. Conversely, if $W_T = k(H_T)$ then $k(x) = F^{-1}(G(x))$ must hold where F is the distribution function of W_T (W_T is cost-efficient). Since $k(x)$ is strictly convex the DARA property holds (Theorem 8). \square

Remark 1. *Consider an investor with optimal terminal wealth $W_T \sim F$. If F has right bounded support then the investor does not exhibit DARA.*

Proof. Recall that $k(x) := F^{-1}(G(x))$ is non-decreasing. Since F is right bounded there exists $b \in \mathbb{R}$, $k(x) \leq b$ ($x \in \mathbb{R}$). We need to show that $k(x)$ cannot be strictly convex ($x \in \mathbb{R}$). We proceed by contradiction, so let $k(x)$ be strictly convex. This implies that it stays above its tangents. As it is non constant and non-decreasing, there exists a point x for which the tangent has positive slope and thus the tangent goes to infinity at infinity. It is thus impossible for k to be bounded from above. \square

5.3 The case of a Black-Scholes market

In a Black-Scholes market we find from (20) that H_T is normally distributed, with mean $rT + \frac{\theta^2 T}{2}$ and variance $\theta^2 T$. It is then straightforward that,

$$\begin{aligned}\mathcal{A}(x) &= \theta f(x) \sqrt{2\pi T} \exp\left(\frac{1}{2} [\Phi^{-1}(1 - F(x))]^2\right) \\ \mathcal{R}(x) &= \theta x f(x) \sqrt{2\pi T} \exp\left(\frac{1}{2} [\Phi^{-1}(1 - F(x))]^2\right).\end{aligned}\quad (35)$$

The financial market thus influences the risk aversion coefficient $\mathcal{A}(x)$ through the instantaneous Sharpe ratio $\theta = \frac{\mu - r}{\sigma}$. Interestingly, the effect of the financial market on the risk aversion coefficient $\mathcal{A}(x)$ is proportional and does not depend on available wealth x . This also implies that in a Black-Scholes market, the properties of the function $x \rightarrow \mathcal{A}(x)$ are solely related to distributional properties of final wealth and do not depend on the precise market conditions. This observation is implicit in the following theorem.

Theorem 10 (DARA in a Black-Scholes market). *Consider an investor who targets some cdf F for his terminal wealth. In a Black-Scholes market the investor has decreasing absolute risk aversion if and only if,*

$$\frac{f(F^{-1}(p))}{\phi(\Phi^{-1}(p))} \text{ is strictly decreasing on } (0, 1), \quad (36)$$

or equivalently,

$$F^{-1}(\Phi(x)) \text{ is strictly convex on } \mathbb{R}, \quad (37)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the density and the cdf of a standard normal distribution.

Proof. Since $G(x)$ is the distribution function of a normally distributed random variable, it is straightforward that $\mathcal{A}(x)$ is decreasing if and only if (36) holds true. Furthermore, $F^{-1}(G(x))$ is strictly convex if and only if $F^{-1}(\Phi(x))$ is strictly convex. The second part of the theorem now follows from Theorem 8. \square

When F is the distribution of a normal random variable, the ratio (36) becomes constant. This confirms the well-known fact that the demand for a normally distributed final wealth in a Black-Scholes market is tied to a constant absolute risk aversion (see also Section 6.2 for a formal proof). In a Black-Scholes setting, the property of DARA remains invariant under changing conditions for the financial market. This is however not true in a general market where risk aversion, demand for a particular distribution, and properties of the financial market are intertwined.

If the final wealth W has a lognormal distribution F then one has, $F^{-1}(\Phi(x)) = \exp(x)$. Since the exponential function is clearly convex, Theorem 10 implies that in a Black-Scholes market the demand for a lognormal distribution corresponds to DARA preferences. In the following proposition, we show that the exponential distribution also corresponds to DARA preferences in this setting. To this end, we only need to show that the survival function $1 - \Phi(x)$ of a standard normal random variable is log-concave. This is well known in the literature and can be seen as a direct consequence of a more general result attributed to Prékopa¹². Here we provide a direct proof.

Theorem 11 (Exponential Distribution exhibits DARA). *In a Black-Scholes market, an expected utility maximizer with exponentially distributed terminal wealth has decreasing absolute risk aversion.*

Proof. Consider an exponentially distributed random variable with distribution $F(x) = 1 - e^{-\lambda x}$ and density $f(x) = \lambda e^{-\lambda x}$, with parameter $\lambda > 0$.

¹²Prékopa (1973) shows that differentiability and log-concavity of the density implies log-concavity of the corresponding distribution and survival function. It is clear that a normal density is log-concave and thus also its survival function.

We need to show that for all $p \in (0, 1)$,

$$\frac{f'(F^{-1}(p))}{f^2(F^{-1}(p))} < \frac{\phi'(\Phi^{-1}(p))}{\phi^2(\Phi^{-1}(p))}. \quad (38)$$

We find for the left hand side,

$$\frac{f'(F^{-1}(p))}{f^2(F^{-1}(p))} = \frac{-1}{1-p}, \quad (39)$$

whereas for the right hand side,

$$\frac{\phi'(\Phi^{-1}(p))}{\phi^2(\Phi^{-1}(p))} = -\sqrt{2\pi}\Phi^{-1}(p)e^{\frac{[\Phi^{-1}(p)]^2}{2}}. \quad (40)$$

Furthermore, for $x > 0$ it holds that $1 - \Phi(x) < \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} \frac{t}{x} e^{-\frac{t^2}{2}} dt = \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, which allows to conclude for all $0 < p < 1$,

$$\frac{1}{1-p} > \sqrt{2\pi}\Phi^{-1}(p)e^{\frac{[\Phi^{-1}(p)]^2}{2}}. \quad (41)$$

Combining (41) with (39) and (40), we find that condition (38) is satisfied which ends the proof. \square

Consider for all $x \in \mathbb{R}$ such that $F(x) < 1$, the hazard function $h(x) := \frac{f(x)}{1-F(x)}$ which is a well-known device for studying heavy-tailed properties of distributions. The hazard function for an exponentially distributed random variable rate function is clearly constant so that a non-increasing hazard function reflects a distribution that is heavier tailed than an exponentially distributed random variable. It is therefore intuitive that investors exhibiting a demand for distributions with a non-increasing hazard function exhibit DARA preferences. The following theorem makes this precise.

Theorem 12 (Sufficient conditions for DARA). *Consider an investor who targets some cdf F for her terminal wealth. Denote by f its density. If the hazard function $h(x)$ is non-increasing (resp. non-increasing for $x > k$), or equivalently if $1 - F(x)$ is log-convex (resp. log-convex for $x > k$), then the investor has decreasing absolute risk aversion (resp. asymptotic DARA).*

Proof. Denote by $E(x) = 1 - e^{-\lambda x}$ the distribution of an exponentially distributed random variable. Since $F^{-1}(\Phi(x)) = F^{-1}(E(E^{-1}(\Phi(x))))$ and

$E^{-1}(\Phi(x))$ is strictly convex (Theorem 11)) and strictly increasing, it is sufficient to show that $F^{-1}(E(x))$ is strictly increasing and convex on $\mathbb{R}^+ \setminus \{0\}$. Since $h(x)$ is non-increasing it follows that F must have a support of the form $[a, \infty)$ for some $a \in \mathbb{R}$ so that F is strictly increasing on $[a, \infty)$. This implies that $F^{-1}(E(x))$ is strictly increasing on $\mathbb{R}^+ \setminus \{0\}$.

Observe next that $\log(1 - F(x)) = -\lambda k(x)$ where $k(x) := H^{-1}(F(x))$. Since $\log(1 - F(x))$ is convex and decreasing, $k(x)$ is concave and increasing i.e. $k'(x) > 0$ and $k''(x) < 0$. Straight-forward differentiation yields that $k^{-1}(x) = F^{-1}(H(x))$ is convex, which ends the proof. \square

We remark from the proof that a random variable with non-increasing hazard function $h(x)$ must assume values that are almost surely in an interval $[a, \infty)$ where $a \in \mathbb{R}$. A lognormally distributed random variable thus has no non-increasing hazard function (but still satisfies the DARA property).

6 Distributions and Corresponding Utility Functions

We end this paper with a section investigating popular distributions and families of utility functions in a Black-Scholes market. We compute the utility corresponding to popular distributions. For each family of utility, there is an optimal distribution of terminal wealth. Reciprocally for each distribution of terminal wealth we construct a new family of utility for which it is optimal. Some of them can be obtained in closed-form while others can only be expressed as an integral. A table summarizing the results is given at the end of this section. Simultaneously we discuss the optimal distribution of final wealth obtained with popular utility functions such as CRRA, Exponential and HARA utilities.

6.1 Uniform Distribution

Assume that consumption is restricted on $(0, 1)$ and let F be the distribution of a standard uniform random variable¹³. Then from (6) in Theorem 3, a

¹³In this situation, we can understand x as the amount of consumption as a fraction of some fixed amount, say the initial budget of ω_0 .

utility function explaining this distribution writes as,

$$u(x) = \int_{1-x}^1 F_{\xi_T}^{-1}(z) dz \text{ for } 0 < x < 1.$$

Let us recall that $\log(\xi_T)$ is normally distributed with mean parameter $-(r + \frac{\theta^2}{2})T$ and variance $\theta^2 T$, with $\theta = \frac{\mu-r}{\sigma}$. Invoking well-known expressions for the Conditional Tail Expectation of a lognormally distributed random variable, it follows that

$$u(x) = e^{-rT} \Phi \theta \sqrt{T} - \Phi^{-1}(1-x) \text{ for } 0 < x < 1. \quad (42)$$

The coefficient of absolute risk aversion (30) is given by

$$\mathcal{A}(x) = \theta \sqrt{2\pi T} \exp \left(\frac{1}{2} [\Phi^{-1}(1-x)]^2 \right)$$

and the relative risk aversion can then be computed as $\mathcal{R}(x) = x\mathcal{A}(x)$. It is clear that $\mathcal{A}(x)$ is not decreasing and thus that the uniform distribution cannot be explained with a utility function exhibiting the DARA property.

6.2 Normal Distribution and Exponential Utility

Let F be the distribution of a normal random variable with mean M and variance Σ^2 . Then from Theorem 3, the utility function explaining this distribution writes as

$$u(x) = -\frac{\Sigma a}{\theta \sqrt{T}} \exp \left(-\frac{x\theta}{\Sigma} \sqrt{T} \right) \quad (43)$$

where $a = \exp \left(\frac{M\theta\sqrt{T}}{\sigma} - rT - \frac{\theta^2 T}{2} \right)$ and $\theta = \frac{\mu-r}{\sigma}$. This is essentially the form of an exponential utility function with constant absolute risk aversion

$$\mathcal{A}(x) = \frac{\theta \sqrt{T}}{\Sigma}.$$

Note that the absolute risk aversion is constant and inversely proportional to the volatility Σ of the distribution. A higher volatility of the optimal distribution of final wealth corresponds to a lower absolute risk aversion. This is consistent with Dybvig and Wang (2012) who show that lower risk

aversion leads to a larger payoff in the sense of second-order dominance¹⁴. Reciprocally, consider the following exponential utility function defined over \mathbb{R} ,

$$U(x) = -\exp(-\gamma x), \quad (44)$$

where γ is the risk aversion parameter and $x \in \mathbb{R}$. This utility has constant absolute risk aversion $\mathcal{A}(x) = \gamma$. The optimal wealth obtained with an initial budget ω_0 is given by

$$X_T^* = \omega_0 e^{rT} - \frac{\theta}{\gamma\sigma} \left(r - \frac{\sigma^2}{2} \right) T + \frac{\theta}{\gamma\sigma} \ln \left(\frac{S_T}{S_0} \right) \quad (45)$$

where $\theta = \frac{\mu-r}{\sigma}$ is the instantaneous Sharpe ratio for the risky asset S , and thus X_T^* follows a normal distribution $\mathcal{N} \left(\omega_0 e^{rT} + \frac{\theta}{\gamma\sigma} (\mu - r)T, \left(\frac{\theta}{\gamma} \right)^2 T \right)$.

6.3 Lognormal Distribution and CRRA and HARA Utilities

Let F be the distribution of a lognormal random variable $\mathcal{LN} \sim (M, \Sigma^2)$ over \mathbb{R}^+ . We showed in Section 4 that the utility function explaining this distribution is a CRRA utility function having decreasing absolute risk aversion $\mathcal{A}(x) = \frac{\theta\sqrt{T}}{x\Sigma}$. Recall that a CRRA utility investor maximizes the following expected utility of final wealth

$$U(x) = \begin{cases} \frac{x^{1-\eta}}{1-\eta} & \eta \neq 1 \\ \log(x) & \eta = 1 \end{cases} \quad (46)$$

where $\eta > 0$ so that it displays decreasing absolute risk aversion (DARA). The absolute risk aversion is $\mathcal{A}(x) = \frac{\eta}{x}$. The optimal wealth obtained with an initial budget ω_0 is calculated as $[U']^{-1}(\lambda \xi_T)$ where λ is chosen to meet the budget constraint. After some straightforward calculations,

$$X_T^* = \omega_0 e^{rT} e^C \left(\frac{S_T}{S_0} \right)^{\frac{\theta}{\eta\sigma}} \quad (47)$$

where $\theta = \frac{\mu-r}{\sigma}$ is the instantaneous Sharpe ratio for the risky asset S and where $C = -\frac{\theta}{\eta\sigma}(\mu - \frac{\sigma^2}{2})T - \frac{\theta^2 T}{2\eta^2} + \frac{\theta^2 T}{\eta}$. This optimal payoff can be

¹⁴In case of two normal distributions with equal mean, increasing second-order stochastic dominance is equivalent to increasing variance.

interpreted as the final payoff of a constant-mix strategy where one invests a proportion $\frac{\theta}{\eta\sigma}$ of the budget ω_0 in the risky asset, and the rest in the risk-free asset. The optimum X_T^* is clearly lognormally distributed, $X_T^* \sim \mathcal{LN}\left(\ln \omega_0 + \left(r - \frac{\theta^2}{2\eta^2} + \frac{\theta^2}{\eta}\right)T, \left(\frac{\theta}{\eta}\right)^2 T\right)$. The utility function for a HARA utility is a generalization of the CRRA utility. It is given by

$$U(x) = \frac{1-\gamma}{\gamma} \left(\frac{ax}{1-\gamma} + b \right)^\gamma \quad (48)$$

where $a > 0$, $b + \frac{ax}{1-\gamma} > 0$. For a given parametrization, this restriction puts a lower bound on x if $\gamma < 1$ and an upper bound on x when $\gamma > 1$. If $-\infty < \gamma < 1$ then this utility displays DARA. In the case $\gamma \rightarrow 1$, the limiting case corresponds to linear utility. In the case when $\gamma \rightarrow 0$, the utility function becomes logarithmic, $U(x) = \log(x + b)$. Its absolute risk aversion is

$$\mathcal{A}(x) = a \left(\frac{ax}{1-\gamma} + b \right)^{-1} \quad (49)$$

The optimal wealth obtained with an initial budget ω_0 is given by

$$X_T^* = C \left(\frac{S_T}{S_0} \right)^{\frac{\theta}{\sigma(1-\gamma)}} - \frac{b(1-\gamma)}{a}$$

where $C = \frac{\omega_0 e^{rT} + \frac{b(1-\gamma)}{a}}{\exp\left(\frac{\theta}{\sigma(1-\gamma)}\left(r - \frac{\sigma^2}{2}\right)T + \left(\frac{\theta}{1-\gamma}\right)^2 \frac{T}{2}\right)}$. Its cdf is

$$F_{HARA}(y) = \Phi \left(\frac{\ln \left(\frac{y + \frac{b(1-\gamma)}{a}}{C^{\frac{1}{a}}} \right) + \frac{\theta}{\sigma(\gamma-1)} \left(\mu - \frac{\sigma^2}{2} \right) T}{\frac{\theta}{\gamma-1} \sqrt{T}} \right).$$

Observe that the optimal wealth for HARA utility is a lognormal distribution translated by a constant term.

6.4 Exponential Distribution

Consider an exponential distribution with cdf $F(x) = 1 - e^{-\lambda x}$ where $\lambda > 0$. The utility function in (6) in Theorem 3 cannot be obtained in closed-form and does not correspond to a well-known utility. The coefficient of absolute risk aversion is given by

$$\mathcal{A}(x) = \theta \lambda \sqrt{2\pi T} \exp \left(-\lambda x + \frac{1}{2} \left[\Phi^{-1} \left(e^{-\lambda x} \right) \right]^2 \right).$$

In Theorem 11 we have shown that $\mathcal{A}(x)$ is decreasing and thus that the exponential distribution corresponds to a utility function exhibiting DARA.

6.5 Gamma and Log-Gamma (with a shift) Distribution

Consider a gamma distribution with shape $k > 0$ and scale $\alpha > 0$ with pdf and cdf given respectively by

$$f_{\Gamma}(x) = \frac{1}{\alpha^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\alpha}}, \quad F_{\Gamma}(x) = \frac{\gamma\left(k, \frac{x}{\alpha}\right)}{\Gamma(k)},$$

where $\Gamma(k) = \int_0^{+\infty} e^{-t} t^{k-1} dt$ and $\gamma\left(k, \frac{x}{\alpha}\right) = \int_0^{\frac{x}{\alpha}} t^{k-1} e^{-t} dt$. The coefficient of absolute risk aversion is given by

$$\mathcal{A}(x) = \frac{\theta x^{k-1} \sqrt{2\pi T}}{\alpha^k \Gamma(k)} \exp\left(-\frac{x}{\alpha} + \frac{1}{2} \left[\Phi^{-1}\left(1 - \frac{\gamma\left(k, \frac{x}{\alpha}\right)}{\Gamma(k)}\right) \right]^2\right). \quad (50)$$

Consider then a log-gamma distribution with shape $k > 0$ and scale $\alpha > 0$. Its pdf and cdf are respectively equal to $f_{L\Gamma}(x) = f_{\Gamma}(\ln x)/x$ and $F_{L\Gamma}(x) = F_{\Gamma}(\ln x)$. The coefficient of absolute risk aversion is given by

$$\mathcal{A}(x) = \frac{\theta (\ln x)^{k-1} \sqrt{2\pi T}}{x \alpha^k \Gamma(k)} \exp\left(-\frac{\ln y}{\alpha} + \frac{1}{2} \left[\Phi^{-1}\left(1 - \frac{\gamma\left(k, \frac{\ln y}{\alpha}\right)}{\Gamma(k)}\right) \right]^2\right).$$

Bagnoli and Bergstrom (2005) show that for $0 < k < 1$, $1 - F_{\Gamma}(x)$ is log-convex (see also their Table 3). Theorem 12 thus implies that in this instance the Gamma distribution corresponds to DARA preferences. When $k \geq 1$, $1 - F_{\Gamma}(x)$ becomes log-concave and the conditions of Theorem 12 are no longer satisfied. However, detailed numerical inspections strongly suggest that $F_{\Gamma}^{-1}(\Phi(x))$ is strictly convex for all $k > 0$ and thus the DARA property appears to hold true in general. Finally, it is clear that $F_{\Gamma}^{-1}(\Phi(x))$ is strictly convex if and only if $F_{L\Gamma}^{-1}(\Phi(x))$ is strictly convex. Hence similar conclusions regarding the DARA property can be made for Log-Gamma distributed random variables.

6.6 Pareto Distribution

Consider a Pareto distribution with scale $m > 0$ and shape $\alpha > 0$, defined on $[m, +\infty)$. Its pdf is $f(x) = \alpha \frac{m^{\alpha}}{x^{\alpha+1}} \mathbb{1}_{x \geq m}$. The coefficient of absolute risk

aversion for $x \geq m$ is given by

$$\mathcal{A}(x) = \frac{\theta \alpha m^\alpha \sqrt{2\pi T}}{x^{\alpha+1}} \exp \left(\frac{1}{2} \left[\Phi^{-1} \left(\left(\frac{m}{x} \right)^\alpha \right) \right]^2 \right).$$

Bergstrom and Bagnoli (2005) show that $1 - F(x)$ is log-convex. Theorem 12 thus implies that the Pareto distribution corresponds to DARA preferences.

6.7 Gumbel Distribution

Consider a Gumbel distribution with scale $\beta > 0$ and location $\mu > 0$ defined on \mathbb{R} . Its pdf and cdf are respectively

$$f(x) = \frac{1}{\beta} \exp \left(-\frac{x - \mu}{\beta} - e^{-\frac{x - \mu}{\beta}} \right), F(x) = \exp \left\{ -e^{-\frac{x - \mu}{\beta}} \right\}.$$

The coefficient of absolute risk aversion is given by

$$\mathcal{A}(x) = \frac{\theta \sqrt{2\pi T}}{\beta} \exp \left(-\frac{x - \mu}{\beta} - e^{-\frac{x - \mu}{\beta}} + \frac{1}{2} \left[\Phi^{-1} \left(1 - \exp \left\{ -e^{-\frac{x - \mu}{\beta}} \right\} \right) \right]^2 \right).$$

Unfortunately, $1 - F(x)$ is log-concave (see Bergstrom and Bagnoli (2005)) and one cannot invoke Theorem 12 to show that the Gumbel distribution corresponds to DARA preferences. Nevertheless, a numerical analysis shows that $F^{-1}(\Phi(x))$ is strictly convex suggesting that DARA holds indeed.

6.8 Fréchet Distribution

Assume that consumption is restricted on $(m, +\infty)$ and follows a Fréchet distribution with scale $s > 0$, shape $\alpha > 0$ and minimum of $m \in \mathbb{R}$. Its pdf and cdf are respectively

$$f(x) = \frac{\alpha}{s} \left(\frac{x - m}{s} \right)^{-1-\alpha} e^{-(\frac{x-m}{s})^{-\alpha}}, F(x) = e^{-(\frac{x-m}{s})^{-\alpha}}$$

The coefficient of absolute risk aversion is given by

$$\mathcal{A}(x) = \frac{\theta \alpha \sqrt{2\pi T}}{s} \left(\frac{x - m}{s} \right)^{-1-\alpha} \exp \left\{ -\left(\frac{x - m}{s} \right)^{-\alpha} + \frac{1}{2} \left[\Phi^{-1} \left(1 - e^{-(\frac{x-m}{s})^{-\alpha}} \right) \right]^2 \right\}$$

Also in this case there is strong numerical evidence that $F^{-1}(\Phi(x))$ is strictly convex suggesting that DARA property holds indeed.

6.9 Burr Distribution

Consider a Burr distribution with parameters $c > 0$ and $k > 0$. Its pdf and cdf are respectively

$$f_B(x) = ck \frac{x^{c-1}}{(1+x^c)^{k+1}}, \quad F_B(x) = 1 - (1+x^c)^{-k}$$

The coefficient of absolute risk aversion is given by

$$\mathcal{A}(x) = \frac{\theta ck x^{c-1}}{(1+x^c)^{k+1}} \sqrt{2\pi T} \exp \left\{ \frac{1}{2} \left[\Phi^{-1}((1+x^c)^{-k}) \right]^2 \right\}$$

In this case, numerical experiments indicate that the DARA property holds only for certain choice of values for the parameters c and k . For example, DARA holds for $(c, k) = (2, 3)$ but not for $(5, 10)$.

6.10 Summary

We summarize the results in Table 1.

Distribution	DARA	CARA
$U(a, b)$		
$N(A, B^2)$		✓
$LN(A, B^2)$	✓	
$\mathcal{E}(\lambda)$	✓	
*Gamma(θ, k)	✓	
*LGamma(θ, k)	✓	
Pareto(α, m)	✓	
*Gumbel(μ, β)	✓	
*Fréchet(α, m, s)	✓	
*Burr(c, k)		

Table 1: The rows of the table refer respectively to the Uniform, Normal, Lognormal, exponential, Pareto, Gumbel, Fréchet and Burr distributions. CARA and DARA respectively stand for Constant (resp. Decreasing) absolute risk aversion. *Evidence is numerical only.

7 Conclusions

Our work offers a new perspective on behavioral finance and optimal portfolio selection. We propose a bridge between the utility function and the distribution of final wealth. This allows us to explain observed investment choices using expected utility theory with a concave utility for all investors with preferences satisfying FSD. Our results can also be employed to perform non parametric estimation of utility using the investor's choice of distribution of final wealth and market data to capture the pricing kernel ξ_T in a non-parametric way (Aït-Sahalia and Lo (2001)).

To find the optimal investment decision for preferences respecting first-order stochastic dominance, it is enough to consider the expected utility setting with a concave non-decreasing utility. The expected utility setting appears thus to be a very general framework in the context of portfolio selection. As illustrated in the last section, our techniques allow to construct new families of utility functions and can also be used to better understand less studied utility functions, such as the newly proposed SAHARA utility by Chen et al. (2011).

Our results may also indicate that portfolio selection in a decision framework which satisfies FSD (equivalently with a law-invariant non-decreasing objective function) has its deficiencies. For instance, Bernard, Boyle and Vanduffel (2011) show that a standard put contract can be substituted by a suitable derivative contract yielding the same distribution at a strictly lower cost but offering no protection against a market crash. The use of standard put contracts then suggests that first-order stochastic dominance is too restrictive and that background risk as a source of state-dependent preferences comes into play when investing. As another example, insurance contracts can be substituted by financial contracts that have the same distribution but are cheaper (Bernard and Vanduffel (2012)). Insurance contracts provide protection against some certain losses, and thus present more value for customers than similarly distributed financial payoffs. Future research includes considering frameworks for which first stochastic dominance can be violated, for example considering ambiguity on the pricing kernel, Almost Stochastic Dominance (ASD) (Levy (2006), chapter 13) or state-dependent preferences (Bernard, Boyle and Vanduffel (2011)).

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